A geometrical approach to find the preferred intonation of chords

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Abstract

Previous work suggested that convexity in the Euler lattice can be interpreted in terms of consonance [1]. In this paper, a second hypothesis is presented that states that compactness in the Euler lattice is an indication of consonance. The convexity and compactness of chords is used as the basis of a model for the preferred intonation of chords in isolation (without a musical context). It is investigated if, and to which degree convexity and compactness are in agreement with the preferred intonation of chords in isolation. As measure of consonance to compare the model to, Euler's Gradus function is used. It is stressed however, that in the context of this paper, Euler's consonance model is able to represent a general consonance model rather than only the Gradus function itself. First, the diatonic chords are observed, after which the compactness, convexity and consonance according to Euler, is calculated for all chords in general containing 2, 3 and 4 notes within a bounded note name space, such that the relation between these three measures can be obtained. The principle of compactness turns out to be a strong indicative of consonance for chords, having the preference over other consonance models that it is simple and intuitive to use.

Keywords: Intonation, chords, compactness, convexity, Euler, consonance

1 Introduction: Tuning of chords in isolation

Many authors have agreed that the intonation of musical tones in Western music can be split in two or more categories (see for example [2, 3, 4]), and their theories agree on the fact that one category accounts for the intonation that is concerned with vertical musical sounds, that is, sounds without any musical context. This type of intonation is referred to by Terhardt [2] as sensory consonance[1], and by Fyk [3] as harmonic tuning. According to Palisca and Moore [5], sensory consonance refers to the immediate perceptual impression of a sound as being pleasant or unpleasant, and may be judged for sounds in isolation (without a musical context) and by people without musical training. In this paper we will focus on this type of intonation applied to chords, which we will refer to as 'consonance'.

Musicians that are not limited to the fixed equal tempered system, such as singers and string players, have some freedom in intonation when playing a chord. These musicians can vary the tones of the chord in order to sing or play every chord as consonant as possible. Until now, no generally accepted intonation theory has been presented that prescribes how to tune chords in isolation. Let us for example consider the interval of a minor seventh. This interval can be composed from a perfect fifth and a minor third or composed from two perfect fourths, giving rise to two different tunings of the interval of a seventh which leads to (at least) two different tunings of a chord containing the interval of a minor seventh. The question we will address in the paper is how to tune chords in isolation, i.e. which of the many possibilities of tunings of a chord to choose.

The intonation of musical tones can be characterized by frequency ratios. Choosing a reference tone, for example C, that is related to the frequency ratio 1, the perfect fifth $C-G$ for example can be associated with the frequency ratio 3/2. One way to identify the ratio belonging to a specific interval is to look at the harmonic series, as shown in figure 1. Another way to identify the ratio belonging to a specific interval is by looking at a

[1] According to Terhardt [2], sensory consonance consists of roughness, sharpness and tonalness. Here, we do not consider these explanations of sensory consonance but only address the fact that sensory consonance applies to chords without a musical context.

Figure 1: A piece of the harmonic series on C. The interval of a perfect fifth can be found between the second and third harmonic, therefore, the frequency ratio associated with this interval reads $3/2$.

tone space that is referred to as the 'Euler lattice'[2], a piece of which is shown in figure 2. This lattice can be constructed using the following projection [12]:

$$\phi : 2^p(3/2)^q(5/4)^r \mapsto (q, r), \qquad p, q, r \in \mathbb{Z} \tag{1}$$

that maps a frequency ratio to a point in a two dimensional space (fig. 2). The frequency ratio 2 corresponds to the interval octave, the ratio $3/2$ corresponds to the perfect fifth, and the ratio $5/4$ corresponds to the ratio major third. To create the Euler lattice (fig. 2), one representative that lies within one octave is chosen from the many frequency ratios that all map onto one point (q, r). For example, the frequency ratios $6/2$, $3/2$, and $3/4$ all map onto the point $(1, 0)$ but only the ratio $3/2$ is displayed in the figure. The resulting figure only includes frequency ratios within one octave, that is, in the interval $[1, 2)$.[3].

Furthermore, every frequency ratio from the resulting figure can be built from a number of perfect fifths (3/2) and major thirds (5/4). For more information, see [13] and [1]. By choosing a reference tone (C in this case)

	25/18	25/24	25/16	75/64	225/128		
40/27	10/9	5/3	5/4	15/8	45/32	135/128	
32/27	16/9	4/3	1	3/2	9/8	27/16	81/64
256/135	64/45	16/15	8/5	6/5	9/5	27/20	81/80
	256/225	128/75	32/25	48/25	36/25	27/25	

	F#	C#	G#	D#	A#		
G	D	A	E	B	F#	C#	
Eb	Bb	F	C	G	D	A	E
Cb	Gb	Db	Ab	Eb	Bb	F	C
	Ebb	Bbb	Fb	Cb	Gb	Db	

Figure 2: Euler lattice represented in frequency ratios and note names respectively. In the right figure, the C has been chosen as reference tone. We refer to these lattices with the terms frequency ratio space and note name space.

and adding perfect fifths and major thirds according to eq. 1, we can now identify every ratio with a note name which results in the right hand lattice of fig. 2. Projecting the two lattices from figure 2 onto each other, the frequency ratio belonging to every interval (with C as base note) can be found.

Coming back to our argument, for some chords, like for example a major triad, the intonation may be clear, but for others there is no consensus. Consider for example a dominant seventh chord $C - E - G - B\flat$. It can be tuned choosing the ratios: $1, 5/4, 3/2, 9/5$ such that the minor seventh is tuned as minor third $6/5$ above the fifth; or tuned as $1, 5/4, 3/2, 16/9$ such that the minor seventh is chosen to be two fourths above the root, and many other possibilities exist.

Regener [14] stated the ambiguity involving just intonation frequency ratios as follows: Each notated interval actually corresponds to an infinite number of frequency ratios, since multiplication of a frequency ratio by any integer power of $81/80$ leaves the notated interval unchanged. Regener [14] describes furthermore two criteria that are commonly used or assumed in determining which are the "preferential" frequency ratios in just intonation corresponding to a given interval:

[2] This lattice representation and minor variants of it have been introduced in numerous articles [6, 7, 8, 9, 10, 11] and are known under the names 'harmonic network', 'Euler lattice' and 'Tonnetz'. In this paper, we will use the term Euler lattice or just 'tone space'.

[3] Note that the Euler lattice presented in figure 2 represents only frequency ratios involving the prime factors $2, 3$ and 5, which is referred to as 5-limit just intonation.

1. Preferred ratios are those involving the lower numbers when in lowest terms.
2. Preferred ratios are those that can be derived by linear combination from known preferred values for other intervals (beginning with the ratios 3/2 for a perfect fifth and 5/4 for a major third), possibly with a certain use of intervals in mind from some musical context.

It may be clear that these two criteria are not always in agreement and do not constitute a full intonation theory for chords in isolation.

Many functions have been constructed to measure the consonance of an interval or chord, for example Helmholtz's [6] roughness function, Euler's Gradus Suavitatis [15, 16], Parncutt's [17] pitch distance or Sethares' [18] dissonance curve based on Plomp and Levelt's [19] model. These functions can be used to put musical intervals in an order of most consonant to dissonant. However, few of these functions have been used (and are difficult to use) to decide about different intonations of the same chord. This motivates the research in this paper. We present a model that can easily distinguish between different intonations of the same chord and selects the one that is the most consonant. The correctness of this selection will be checked against an existing consonance model. Note however that, although we are comparing two models, the model we present here is designed for deciding about different intonations of the same chord, while consonance models such as Euler's and Helmholtz's find their use in ranking different intervals on the basis of consonance. Although Euler's model can also be used to find the preferred intonation of a chord (as we will see), it is involves some mathematical calculation for each chord, and the model itself is therefore not ideal for quickly deciding between different intonations of the same chord. The model we will present in this paper on the other hand, will be ideal for this purpose.

This paper is concerned with intonation based on frequency ratios only. This means that the presented intonation model applies to either simple tones or tones whose pitch is determined by the frequency of the fundamental; the influence of other phenomena that apply to the pitch of a tone is not considered here. It has been suggested that timbre influences consonance [20] and therefore the origin of consonance would lie in the instruments of a specific culture. However, string and wind instruments naturally produce a sound that consists of exact multiples of a fundamental frequency. Therefore, the term consonance used in this paper, can be thought of as related to sounds produced by these Western instruments, and thus applies to Western music only.

2 A new model for intonation

2.1 Convexity and compactness in the Euler lattice

Comparing the Euler lattice composed from frequency ratios to the lattice composed from note names, it can be understood that the difference between two ratios having the same note name is a factor 81/80 (or multiples hereof), which is known as the syntonic comma (see figure 3). Thus different locations for a single note name

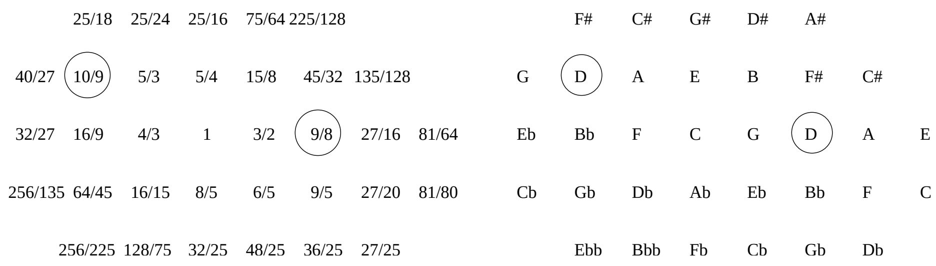
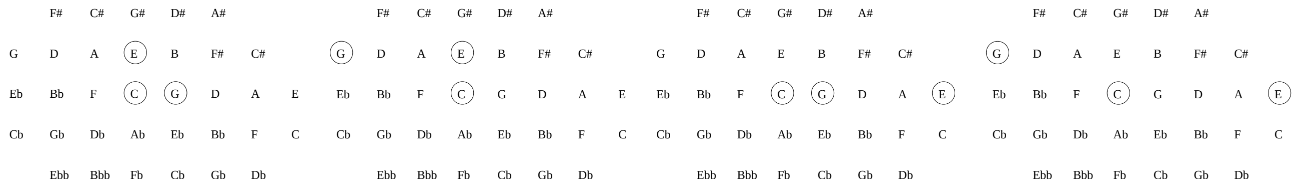


Figure 3: The difference between frequency ratios of the two indicated D's is a factor $(9/8)/(10/9) = 81/80$.

exist in the note name space, all giving rise to a different ratio and therefore different intonation. A chord, defined by a set of note names (such as C, E, G) can be represented by several configurations in the tone space, as we can understand by looking at figure 4. By the configuration of a set, we mean the locations of all elements in this set, indicated in the tone space. Every configuration of the chord represents a different intonation. The configuration of a set of note names can be changed by moving the frequency ratio(s) corresponding to one or more elements of the set by a syntonic comma (=81/80) since the ratio 81/80 is the difference between two ratios with the same note name.



(a) (b) (c) (d)

Figure 4: Possible configurations of the triad C, E, G.

It has been observed that the diatonic major and minor scale as well as all diatonic chords[4] form convex and compact sets in the Euler lattice [1, 12]. Furthermore, it has been explained that convexity in the Euler lattice can be interpreted in terms of consonance [1]. Therefore, we could hypothesize that a convex set represents the preferred intonation. Convexity on a two dimensional lattice has been defined as follows: A set is convex if all elements that lie in the convex hull are included in the set. In other words: A set is convex if, drawing lines between all points in the set, all elements which lie within the spanned area are elements of the set [1]. A set of notes can have more than one convex configuration in the note name space. For example, the two-note set C-G tuned as $1 - 3/2$ is convex, but the tuning as $1 - 40/27$ is also convex. To adjust our hypothesis, in cases like this, a choice has to be made between the two configurations to present the preferred tuning. A possibility is to choose the most compact one. Compactness is intuitively understood as the extend to which elements of a set are close together in a set. In a three dimensional space the most compact object would be shaped like a ball. In this paper, the compactness is calculated by summing the distances between all pairs of points (notes in the tone space); the lower the resulting value, the more compact is the set. As an example, the left most configuration from figure 4 represents the most compact configuration of the chord C, E, G. The decision to choose the most compact set is not a random choice. If two notes are close together in the frequency ratios space, they have many prime factors in common, as the tone space was built from powers of the primes 2,3 and 5 [1]. Therefore, the closer together two notes are in the tone space, the smaller are the integers forming the ratio that represents the interval between the two notes. According to just intonation, ratios with small integer ratios are preferred. Generalizing this for chords consisting of more than two notes, the intonation of a chord whose notes are the most close together in the tone space should be preferred. Now we have motivated why to use compactness to decide which of the possible convex sets represents the preferred intonation, we can actually make two hypotheses:

1. the preferred intonation of a chord is represented by the most compact set of the possible convex configurations of that chord.
2. the preferred intonation of a chord is represented by the most compact configuration of that chord.

Note that these hypotheses contain an empirical component, since "preferred intonation" applies to the perception of humans. However, we will follow here the path of investigating the correlation between the hypotheses and an established consonance measure.

2.2 Euler's Gradus Suavitatis

The hypotheses proposed in the previous section will be applied to a number of chords, and the result will be compared with an existing consonance measure. The consonance measure we will use is Euler's Gradus function, since it applies, similar to the hypotheses, to frequency ratios of chords in isolation. Although nowadays Helmholtz's [6] consonance theory which is based on the beating of partials, seems to be most supported [21, 19, 18, 22, 23], the difference between Helmholtz's and Euler's theory is small in view of our purposes in this chapter. The order from the most to least consonant interval according to Euler or Helmholtz may differ slightly, but the question we will address in this paper is about the most consonant frequency ratios given a chord or interval. From all possibilities, Helmholtz and Euler's theories will choose virtually always one and the same set of frequency ratios. For example, given the interval $C - E$, what is the frequency ratio that makes this interval as consonant as possible? Both Euler and Helmholtz rate 5/4 as the most preferred intonation for this interval.

Euler developed his Gradus Suavitatis (degree of softness) Γ. The function is defined as a measure of the simplicity of a number or ratio. Any positive integer a can be written as a unique product $a = p_1^{e_1} \cdot p_2^{e_2} \dots p_n^{e_n}$ of

[4]Diatonic chords are understood here as all chords that are subsets of a diatonic set. In the minor mode, the chromatic alterations leading to harmonic minor are also taken in account, such that, for example, the augmented triad can also be found as a diatonic chord.

positive integer powers e_i of primes $p_1 < p_2 < \ldots < p_n$. Euler's formula is then defined as:

$$\Gamma(a) = 1 + \sum_{k=1}^{n} e_k(p_k - 1), \tag{2}$$

$\Gamma(a)$ is a number that expresses the simplicity of a. The lower the number the simpler is a. For intervals and chords, a so-called *exponent* needs to be calculated to obtain Γ from. For an interval x/y the exponent is the ordinary product $x \cdot y$ so $\Gamma(x \cdot y)$ expresses the simplicity of the interval x/y. For chords where the frequency ratios are expressed as $a : b : c$, the exponent is given by the Least Common Multiple (LCM) of these a, b and c. The Gradus Suavitatis is then calculated as $\Gamma(LCM(a, b, c))$. Euler connected the simplicity of chords and intervals with the consonance thereof. The lower the value $\Gamma(LCM(a, b, c))$, the more consonant is the chord $a : b : c$. Here is an example to calculate the Gradus Suavitatis. A major triad $1 : 5/4 : 3/2$ can be written as $4 : 5 : 6$ which can in turn be written as $2^2 : 5 : 2 \cdot 3$ (to make the calculation of the LCM easier). The LCM of these numbers is then $2^2 \cdot 3 \cdot 5 = 60$ and the Gradus Suavitatis of 60 is $\Gamma(60) = 1 + 2 \cdot 1 + 1 \cdot 2 + 1 \cdot 4 = 9$. According to the tonal space that maps frequency ratios to note names (see figure 2), this chord can also be tuned differently, for example as $1 : 5/4 : 40/27$, the fifth of the triad is then changed by the syntonic comma $(81/80)$: $\frac{3}{2}/\frac{81}{80} = \frac{40}{27}$. The ratios $1 : 5/4 : 40/27$ can be written differently as $108 : 135 : 160 = 2^2 \cdot 3^3 : 3^3 \cdot 5 : 2^5 \cdot 5$. Then the LCM equals $2^5 \cdot 3^3 \cdot 5 = 4320$ which results in $\Gamma(4320) = 16$. This is obviously higher than the value for the $4 : 5 : 6$ chord and this means that this chord is less consonant than the $4 : 5 : 6$ chord according to this function.

2.3 Configurations in the tone space indicating the intonation

In the same way we can compare other chords in several tunings to see which tuning is most preferable. We compare different configurations of a chord. As we have seen, the configuration of a set of note names can be changed by moving one or more elements of the set by a syntonic (=81/80). In the tables 1, 2, and 3, the diatonic chords are listed with a number of possibilities for tuning.

3-note chords	convex	
major triad C-E-G	$1 - 5/4 - 3/2$ · ⊙ · · ● ⊙ · · · $\Gamma = 9$	$1 - 5/4 - 40/27$ ⊙ · · ⊙ · · · ● $\Gamma = 16$
minor triad C-E♭-G	$1 - 6/5 - 3/2$ · · · · ● ⊙ · · ⊙ $\Gamma = 9$	$1 - 32/27 - 3/2$ · · · · · ⊙ · · ● ⊙ · · · · · $\Gamma = 15$
diminished triad C-E♭-G♭	$1 - 6/5 - 36/25$ ● · · · ⊙ · · · ⊙ $\Gamma = 15$	$1 - 32/27 - 64/45$ ⊙ · · ● · ⊙ · · $\Gamma = 17$
augmented triad C-E-G♯	$1 - 5/4 - 25/16$ · ⊙ · · ⊙ · · ● · $\Gamma = 13$	$1 - 5/4 - 125/81$ ⊙ · · · · · · · · · · · · · ⊙ · · · · ● $\Gamma = 23$

Table 1: Diatonic chords consisting of 3 notes. Chords like $C - E - A$ and $C - F - A$ have been omitted as duplications of the root-position forms. Of each chord, the convex configuration is given, together with another possible configuration. More configurations (intonations) are possible but only one is given here. The circles represent the notes in the frequency ratio space, the black circle representing the root C of the chord.

Since the tone space is infinitely big, there are infinitely many tunings for a chord, however only some musically logical ones are listed here to give an example. In the first column of every table the name of the chords with

4-note chords	convex		
dominant seventh chord C-E-G-B♭	1:5/4:3/2:9/5 ⊙ · · ● ⊙ · · · ⊙ $\Gamma = 15$	1 : 5/4 : 3/2 : 16/9 · · ⊙ · ⊙ · ● ⊙ · · · · $\Gamma = 17$	
major seventh chord C-E-G-B	1 : 5/4 : 3/2 : 15/8 · ⊙ ⊙ · ● ⊙ · · · $\Gamma = 10$	1 : 5/4 : 3/2 : 50/27 ⊙ · · · · · · · ⊙ · · · · ● ⊙ $\Gamma = 18$	
minor seventh chord C-E♭-G-B♭	1 : 6/5 : 3/2 : 9/5 · · · ● ⊙ · · ⊙ ⊙ $\Gamma = 11$	1 : 6/5 : 3/2 : 16/9 ⊙ · ● ⊙ · · · ⊙ $\Gamma = 16$	
half-diminished seventh chord C-E♭-G♭-B♭	1 : 6/5 : 36/25 : 9/5 ● · · · ⊙ ⊙ · · ⊙ $\Gamma = 15$	1 : 6/5 : 36/25 : 16/9 ⊙ · ● · · · · · ⊙ · · · · · ⊙ $\Gamma = 19$	1 : 6/5 : 64/45 : 16/9 ⊙ · ● · ⊙ · · ⊙ $\Gamma = 17$
major-minor seventh chord C-E♭-G-B	1 : 6/5 : 3/2 : 15/8 · ⊙ ● ⊙ · ⊙ $\Gamma = 15$	1 : 6/5 : 3/2 : 50/27 ⊙ · · · · · · · · · · · · ● ⊙ · · · · ⊙ $\Gamma = 23$	
augmented seventh chord C-E-G♯-B	1:5/4:25/16:15/8 ⊙ · ⊙ ⊙ ● · $\Gamma = 15$	1:5/4:125/81:50/27 ⊙ · · · · · ⊙ · · · · · · · ⊙ · · · · ● $\Gamma = 25$	1:5/4:25/16:50/27 ⊙ · · ⊙ · · · ⊙ · · · ● $\Gamma = 20$
diminished seventh chord C-E♭-G♭-B♭♭	1:6/5:36/25:216/125 ● · · · · ⊙ · · · · ⊙ · · · · ⊙ $\Gamma = 22$	1:6/5:64/45:128/75 · · ● · ⊙ · · ⊙ · ⊙ · · $\Gamma = 22$	1:6/5:36/25:128/75 · · · · · ● · · · · ⊙ · ⊙ · · ⊙ $\Gamma = 22$
major triad with added sixth C-E-G-A	1 : 5/4 : 3/2 : 5/3 ⊙ ⊙ · · ● ⊙ $\Gamma = 11$	1 : 5/4 : 3/2 : 27/16 ⊙ · · · ● ⊙ · ⊙ $\Gamma = 13$	
minor triad with added sixth C-E♭-G-A♭	1:6/5:3/2:8/5 · · · · ● ⊙ · ⊙ ⊙ $\Gamma = 11$	1:6/5:3/2:81/50 ● ⊙ · · · · ⊙ · · · · · · · ⊙ $\Gamma = 19$	

Table 2: Diatonic chords consisting of 4 notes.

corresponding note names is given. In the other columns different tunings and their configurations in the plane are given. The tones are indicated by circles, the black circle being the root of the chord (C was chosen to be the root in all cases). For every chord, the Gradus Suavitatis is calculated and given in the tables. We can test our first hypothesis which says that the convex configuration (and if there is ambiguity, the most compact convex configuration) represents the preferred intonation. One can see that for almost every chord the convex configuration of it in the tone space is more consonant according to Euler (i.e., lower value for Γ) than the other. There are two exceptions to this which are the diminished seventh chord and the dominant eleventh chord. The

5/6/7-note chords	convex	
dominant ninth chord C-E-G-B♭-D	1:5/4:3/2:9/5:9/8 ⊙ · · ● ⊙ ⊙ · · ⊙ $\Gamma = 16$	1:5/4:3/2:16/9:10/9 ⊙ · ⊙ · ⊙ · ● ⊙ $\Gamma = 17$
dominant eleventh chord C-E-G-B♭-D-F	1:5/4:3/2:9/5:9/8:27/20 ⊙ · · · ● ⊙ ⊙ · · · ⊙ ⊙ $\Gamma = 18$	1:5/4:3/2:16/9:10/9:4/3 ⊙ · ⊙ · ⊙ ⊙ ● ⊙ · · · · $\Gamma = 17$
dominant thirteenth chord C-E-G-B♭-D-F-A	1 : 5/4 : 3/2 : 9/5 : 9/8 : 27/20 : 27/16 ⊙ · · · ● ⊙ ⊙ ⊙ · · ⊙ ⊙ $\Gamma = 19$	1 : 5/4 : 3/2 : 16/9 : 10/9 : 4/3 : 5/3 ⊙ ⊙ ⊙ · ⊙ ⊙ ● ⊙ · · · · also convex! $\Gamma = 17$

Table 3: Diatonic chords consisting of 5, 6 or 7 notes.

diminished seventh chord can be tuned in various ways to give the same 'consonance value'. This can perhaps be explained from the fact that this chord is a reasonably dissonant chord and therefore changing one of the intervals by a comma (81/80) has less impact than doing this with a more consonant chord[5].

The dominant eleventh chord is in a sense a reduction of the dominant thirteenth chord, only one note is missing (see table 3). Filling in the missing note in the configuration that is most favored (according to Euler), one obtains the most consonant thirteenth chord which is convex as well. In this way, we can understand why this particular configuration for the eleventh chord is more consonant than the convex one. However, this second configuration is more compact than the first one, supporting hypothesis number 2, which says to prefer the most compact configuration. Hypothesis no. 2 was also validated in all other cases except for the diminished seventh chord. The second configuration of the diminished seventh chord listed in table 2 is the most compact one. The compactness of a configuration may be difficult to judge at first sight. In the next section we will present a mathematical formula to calculate compactness. We checked (with a Matlab program, as we will see) all other tuning possibilities (in a sensible range) of these chords by multiplying one or more of the ratios with $(81/80)^n$ and verifying whether this resulted in a lower value for Γ. Note that both listed dominant thirteenth chords are convex. The second one listed is the preferred one according to both hypotheses, since it is more compact. This is also the configuration which is preferred by Euler's function.

To sum up, we proposed two hypotheses in order to present the best intonation, the first saying to prefer (the most compact) convex configuration, and the second saying to prefer the most compact configuration. The values of consonance of the chords were calculated using Euler's Gradus function. Of the 16 chords, for 14 of them hypothesis 1 was validated. For 15 of them, hypothesis 2 was validated. The exceptions can be explained from music theory and from the convexity theory itself. In section 4 the correlation between the concepts of consonance, convexity and compactness will be investigated further. Moreover, we want to stress that "preferred tuning" in this case is only based on the sound of the chord in isolation. In musical practice, there can be more than one choice for the intonation of a chord depending on its musical function in the chord sequence. However, this can still be a very useful measure because it can serve as the beginning of a full tuning theory.

3 Mathematical correlation between compactness and Euler's consonance

In the above sections we have shown that the convex and compact configuration of a set of notes may give an indication of the most consonant sound, as predicted by the measure of Euler. In this section we will try to

[5]The syntonic comma $81/80 = 3^4/(2^4 \cdot 5)$ has factors of 2 3 and 5 in it. Therefore, if the Least common multiple of a chord is already high, the chance that it changes a lot after one of the intervals is multiplied by a comma is low, since the LCM is constructed by multiplying the highest powers of 2, 3 and 5 form the intervals.

formalize this in a mathematical way. Looking at the formula for Euler's Gradus function (eq. 2), we understand that the value for Γ becomes bigger when there are more factors of $2, 3$ and 5 in the LCM of the chord. If two notes are close together in the tone space, they have many prime factors in common, as explained in the previous section. This suggests that the Gradus function is related to the compactness of a set. We will therefore try to formalize the relation between compactness and consonance according to Euler's function.

Since our 2-dimensional tonal space (figure 2) neglects all factors of 2, we first consider the 3-dimensional tonal space that allows also all octave transpositions. In this way, we consider all axes representing the powers of primes, that is the x-axis representing the powers of 2, the y-axis representing the powers of 3 and the z-axis representing the powers of 5. In this coordinate system, a point with coordinates (a, b, c), represents the frequency ratio $2^a \cdot 3^b \cdot 5^c$.

3.1 Compactness in 3D

We want to consider a set of points (representing a chord) in the 2-3-5-space (octave-fifth-third-space) and measure its compactness. Several definitions of compactness are possible. The most intuitive way to measure compactness for our purposes is to sum the distances between all pairs of points[6]. The lower the value of the sum, the more compact the set is. The compactness C of a set of notes is then defined as follows:

$$C = \sum_{1 \leq i,j \leq n} |\vec{x_i} - \vec{x_j}| = \sum_{1 \leq i,j \leq n} \sqrt{(x_{i1} - x_{j1})^2 + (x_{i2} - x_{j2})^2 + (x_{i3} - x_{j3})^2}, \tag{3}$$

where $\vec{x} = (x_1, x_2, x_3)$ defines the coordinates of a tone in the tone space. The term $x_{i1} - x_{j1}$ now defines the difference in the factor 2, the $x_{i2} - x_{j2}$, the difference in the factors 3, and $x_{i3} - x_{j3}$ the difference in the factors 5.

Each tone (note name) has more than one position in the tone space, which means that each chord has several configurations in tone space as we saw in the previous section. The factor that changes the frequency ratio but keeps the note name constant is $81/80 = 2^{-4}3^4 5^{-1}$. Multiplying a frequency ratio by this factor means moving a point in the $2, 3, 5$-coordinate system over $(-4, 4, -1)$. Given a set of points $\vec{x_1}\vec{x_2}\ldots\vec{x_n}$, every $\vec{x_i}$ has a number of possible coordinates such that the point represents the same note name:

$$\vec{x_i} = \begin{pmatrix} x_{i1} - 4k_i \\ x_{i2} + 4k_i \\ x_{i3} - k_i \end{pmatrix}, \qquad k \in \mathbb{Z}. \tag{4}$$

The compactness C can thus be written as:

$$\begin{aligned} C &= \sum_{1 \leq i,j \leq n} \sqrt{X_1 + X_2 + X_3}, \\ X_1 &= (x_{i1} - x_{j1} - 4(k_i - k_j))^2 \\ X_2 &= (x_{i2} - x_{j2} + 4(k_i - k_j))^2 \\ X_3 &= (x_{i3} - x_{j3} - (k_i - k_j))^2 \end{aligned} \tag{5}$$

and the most compact configuration of a set $\vec{x_1}\vec{x_2}\ldots\vec{x_n}$ is given by the $k_2, k_3, \ldots, k_n$ for which C has a minimum[7].

The value for Euler's Gradus function can now be calculated for a certain configuration of points. Therefore, we first need to find the Least Common Multiple (LCM) of the chord. To be able to find the LCM of a chord we have to write the chord in the form $a : b : c$ such that a, b, c are integers (just like we did in section 2 were the chord $1 : 5/4 : 3/2$ was written as $4 : 5 : 6$). Since the point x_{ij} represents the j^{th} coordinate (meaning the multiples of 2,3 or 5) of note x_i, a whole frequency ratio is expressed as $2^{x_{i1}-4k_i} \cdot 3^{x_{i2}+4k_i} \cdot 5^{x_{i3}-k_i}$. A (3-note) chord $a : b : c$ can therefore be written as

$$\begin{aligned} a : b : c = \; & 2^{x_{11}-4k_1} \cdot 3^{x_{12}+4k_1} \cdot 5^{x_{13}-k_1} : \\ & 2^{x_{21}-4k_2} \cdot 3^{x_{22}+4k_2} \cdot 5^{x_{23}-k_2} : 2^{x_{31}-4k_3} \cdot 3^{x_{32}+4k_3} \cdot 5^{x_{33}-k_3}. \end{aligned} \tag{6}$$

[6]Note that this concept of compactness is different from the concept of a 'compact set' in topology.

[7]One of the k_i is fixed (k_1 in this case) and set to zero because the set needs to have a reference point. If all k_i were to be chosen freely, many sets with the same compactness but different locations may exist.

If the chord is already in a form such that a, b, c are (positive) integers, the LCM of 6 can be found as follows:

$$\mathrm{LCM}(a,b,c) = 2^{\max\{x_{11}-4k_1,\ldots,x_{n1}-4k_n\}} \cdot 3^{\max\{x_{12}+4k_1,\ldots,x_{n2}+4k_n\}} \cdot 5^{\max\{x_{13}-k_1,\ldots,x_{n3}-k_n\}} \tag{7}$$

where $\max\{a_1, ..., a_n\}$ picks the largest of numbers a_1 to a_n.

When a, b, c are not integers, the expression for the LCM looks a bit different. To write the chord $a : b : c$ in a form such that it is represented by integers, a, b, c should be multiplied by the Least Common Multiple (LCM) of the denominators of a, b, c (for example to write the chord $1 : 5/4 : 3/2$ as $4 : 5 : 6$ each ratio was multiplied by $\mathrm{LCM}(1, 2, 4) = 4$). The fact that a, b and c are split into powers of $2, 3$ and 5 makes this process easier. Instead of finding the LCM of the denominators we just need to find the maximum of all factors of $2, 3$ and 5 in the denominators. The LCM of 6 then changes as follows:

$$\mathrm{LCM}(a,b,c) = 2^{v_1} 2^{\max\{x_{11}-4k_1,\ldots,x_{n1}-4k_n\}} \cdot 3^{v_2} 3^{\max\{x_{12}+4k_1,\ldots,x_{n2}+4k_n\}} \cdot 5^{v_3} 5^{\max\{x_{13}-k_1,\ldots,x_{n3}-k_n\}} \tag{8}$$

where

$$v_j = \max A, \qquad A = \{-z | z \in B_j \,\&\, z < 0,\ j = 1, 2, 3\} \tag{9}$$

and

$$B_j = \bigcup_{1\le i\le n} \{z_{ij}\} \quad \text{where} \quad \begin{cases} z_{i1} = x_{i1} - 4k_i \\ z_{i2} = x_{i2} + 4k_i \\ z_{i3} = x_{i3} - k_i \end{cases} \tag{10}$$

and n is the number of notes in the chord. Finally, the value for Γ (defined in eq. 2) is given as follows:

$$\begin{aligned} \Gamma &= 1 + v_1 + \max\{x_{11} - 4k_1, \ldots, x_{n1} - 4k_n\} \\ &+ 2 * (v_2 + \max\{x_{12} + 4k_1, \ldots, x_{n2} + 4k_n\}) \\ &+ 4 * (v_3 + \max\{x_{13} - k_1, \ldots, x_{n3} - k_n\}). \end{aligned} \tag{11}$$

We now have expressions for C and Γ and we would like to see that the $k_2 \ldots k_n$ that make C minimal, also make Γ minimal, for a correspondence between compactness and consonance. A Matlab program has been written that can calculate the values for C, Γ and the values for k that make both equations minimal. This is done by varying the coordinates of point 2 over all points of the 3-Dim space in which every coordinate runs from -4 to 4. Point 1 is taken at the origin. It turns out that in 86.5% of the cases both C and Γ have a minimum for the same k. For the case $n = 3$ can in the same way also be calculated if the same value for k_i makes the C and Γ minimal. It turns out this is true for 70.1% of the cases. For $n > 3$ the problem becomes computationally very intensive[8]. However, from this we can conclude that the hypothesis: the more compact, the more consonant is true in the majority of the cases.

3.2 Compactness in 2D

In the 2-D space where all chords are projected, the frequency ratios are considered under octave equivalence. Instead of considering the x_{i1}, x_{i2} and x_{i3} component (2,3 and 5 component) we only consider the x_{i2} and x_{i3} component (for convenience we kept these names) in the 2D space. Thus the expression for C becomes simpler:

$$C = \sum_{1\le i,j\le n} \sqrt{(x_{i2} - x_{j2} + 4(k_i - k_j))^2 + (x_{i3} - x_{j3} - (k_i - k_j))^2}. \tag{12}$$

The expression for Γ however, becomes more complicated. The first term $\max\{x_{11} - 4k_1, \ldots, x_{n1} - 4k_n\}$ changes. A point in the plane now only is specified by its 3- and 5-components: $3^{x_{i2}+4k_i} 5^{x_{i3}-k_i}$. The factor 2^n that together with this specifies the whole frequency ratio: $2^n \cdot 3^{x_{i2}+4k_i} \cdot 5^{x_{i3}-k_i}$, only serves to keep the frequency ratio within the interval $[1, 2)$. Therefore, to find an expression for n, we need to solve:

$$1 \le 2^n \cdot 3^{x_{i2}+4k_i} \cdot 5^{x_{i3}-k_i} < 2. \tag{13}$$

[8] The number of possible configurations of a set consisting of n points, increases with n as $\binom{728}{n-1}$, since this expresses the number of possibilities to choose n points from a $9 \times 9 \times 9$ lattice where one note is fixed in the origin.

From this n can be analytically solved as

$$-(x_{i2}+4k_i)\cdot\log_2 3-(x_{i3}-k_i)\cdot\log_2 5\leq n<1-(x_{i2}+4k_i)\cdot\log_2 3-(x_{i3}-k_i)\cdot\log_2 5. \quad (14)$$

Since n should be an integer, this makes:

$$n=\lceil-(x_{i2}+4k_i)\cdot\log_2 3-(x_{i3}-k_i)\cdot\log_2 5\rceil, \quad (15)$$

where $\lceil x\rceil$ is the smallest integer greater or equal to x. We can therefore understand that the first term in Γ can now be replaced by $\max\{\lceil-(x_{12}+4k_1)\cdot\log_2 3-(x_{13}-k_1)\cdot\log_2 5\rceil,\lceil-(x_{22}+4k_2)\cdot\log_2 3-(x_{23}-k_2)\cdot\log_2 5\rceil,\ldots,\lceil-(x_{n2}+4k_n)\cdot\log_2 3-(x_{n3}-k_n)\cdot\log_2 5)\rceil\}$, thus Γ becomes:

$$\begin{aligned}\Gamma &= 1+v_1+\max\{\lceil-(x_{12}+4k_1)\cdot\log_2 3-(x_{13}-k_1)\cdot\log_2 5\rceil,\ldots \\ &\quad \ldots,\lceil-(x_{n2}+4k_n)\cdot\log_2 3-(x_{n3}-k_n)\cdot\log_2 5\rceil\} \\ &+ 2*(v_2+\max\{x_{12}+4k_1,\ldots,x_{n2}+4k_n\}) \\ &+ 4*(v_3+\max\{x_{13}-k_1,\ldots,x_{n3}-k_n\}),\end{aligned} \quad (16)$$

with v_1 now given by:

$$\begin{aligned} v_1 &= \max\{-z|z\in B\,\&\,z<0\}, \\ B &= \bigcup_{1\leq i\leq n}\{\lceil-(x_{i2}+4k_i)\cdot\log_2 3-(x_{i3}-k_i)\cdot\log_2 5\rceil\}\end{aligned} \quad (17)$$

and v_2, v_3 as given in eq 9,10. Using these expressions, we calculate the number of cases for which the value of k that makes C minimal also makes Γ minimal. In table 4 all percentages are given. Surprisingly, the percentages for the 2-D lattice are higher than for the 3-D lattice. Using the Matlab program we have also checked all chords

lattice	number of notes	percentage correct
3-D	2	86.5 %
3-D	3	70.1 %
2-D	2	97.5 %
2-D	3	85.4 %
2-D	4	76.8 %

Table 4: Results of testing the hypothesis: the configuration of a chord that is most compact is also most consonant.

that are listed in tables 1, 2 and 3, to be sure that we indeed listed the most compact configurations in these tables. It indeed turns out that the hypothesis "the configuration that is most compact, is the most consonant according to Euler's value" is true for all chords except for the diminished seventh chord.

3.3 Interpretation of results

How can these results be explained, and can they perhaps be related to convexity? We have tested the hypothesis "the more compact, the more consonant" for all possible 2,3,4-tone sets within a 2-D 9×9 lattice or a 3-D $9\times 9\times 9$ lattice. It turns out that the percentages of correspondence are reasonably high. In the 2-D space we gained a little higher percentage than in the 3D space. One of the reasons that the relationship between consonance and compactness is not a one-to-one correspondence has to do with the weights in the definition of Γ and with the measure of the syntonic comma in the 2-D and 3-D space. The weights $1, 2$ and 4 in the definition for Γ (eq. 11) cause Γ to change more due to a shift in the 3-coordinate than to a shift in the 2-coordinate (factor 2). Γ is changed most due to a shift in the 5-coordinate (factor 4). Therefore we can also understand that the percentages decrease as the number of notes increase: the more notes, the more directions in the lattice are involved. In the expression for C (eq. 5) we see two times a factor 4 (which comes from the syntonic comma $2^{-4}3^4 5^{-1}$) in the terms that are concerned with the distances in the 2 and 3 direction. This means that the compactness C is more influenced by changes in the 2 and 3 direction than by changes in the 5 direction. We therefore understand that there cannot be a one to one correspondence between the compactness C and the consonance measure Γ. Note however that a one-to-one correspondence between the two functions cannot be created by choosing the weights of our compactness function differently, due to the fact that the functions C and Γ differ on many more levels. Furthermore, a reason for not changing the equal weights in our compactness function C, is that compactness is an intuitive measure which can be judged by the eye. If these weights were changed, it would not be that easy anymore to judge the compactness of two sets on first sight.

It was already intuitively clear that consonance Γ could be related to compactness C in a general way. We have seen that for a high percentage of sets, the most consonant configuration is also the most compact one. Imagine a convex and highly compact set in the 3-D space centered around the origin. If one of the elements of the set is moved by the syntonic comma it is moved by the vector $(-4, 4, -1)$ (since $81/80 = 2^{-4}3^{4}5^{-1}$). The new set is always less compact than it was if the size of the set is less than a certain number of elements, since the vector $(-4, 4, -1)$ then takes the element outside the area spanned by the other elements (which makes the set less compact). For a set in the 2-D space the syntonic comma is represented by the vector $(4, -1)$ which yields the same conclusion. Consider now a convex and highly compact set centered in the lattice, and imagine what happens with the consonance Γ if one or more elements are shifted by a syntonic comma. Again if the number of elements is within a certain range, a shift by $81/80 = 2^{-4}3^{4}5^{-1}$ will increase the LCM of the chord and the new chord will be less consonant. Since here we have only observed sets consisting of $2, 3$ and 4 elements it is understandable that shifting one or more elements of the set by a syntonic comma makes the set less compact and less consonant.

Now we want to make a connection to convexity. If we look at table 3, the last column represents chords that are all introduced as alternative intonations of the chords mentioned. It is remarkable that these configurations all have the same value for Γ, namely $\Gamma = 17$. By looking at the configurations, we understand that it does not matter if the inner notes are filled, the value of Γ just depends on the boundary notes. This is understandable since the value of Γ only depends on the Least Common Multiple of the frequency ratios (written in integers as we have shown) instead of depending on all frequency ratios. The LCM picks the highest factors of 2,3 and 5, which precisely indicate the boundaries of the chord. Therefore the value of Γ of a chord equals the value of Γ of the chord that represents the convex hull. The *convex hull* of a set of points S is the intersection of all convex sets containing S. For example, the rightmost chord at the bottom of table 3 represents the convex hull of the two chords above this chord. It can now be seen that whenever there is a possibility of a convex configuration of a chord, this will often be the most compact one. How often that is, we will investigate in the next section.

4 Convexity, compactness and consonance

In the previous experiments we varied the coordinates of a (2-D or 3-D) space to represent sets of notes for which we wanted to calculate whether the most compact configuration corresponded to the most consonant configuration. We now want to know whether these sets do also correspond with a convex configuration. More precisely: which percentage of the sets of notes that have a possible convex configuration, have a convex configuration that corresponds with 1) the most compact configuration, and 2) the most consonant configuration. For some chords, there is no possible intonation such that the notes form a convex set in the tone space. For these chords, only the compactness can say something about the preferred intonation. Figure 5 illustrates what percentages we are looking for. In the figure, the set S consists of all configurations of all possible chords consisting of n notes. The set T consists of all configurations of the chords that have a possible convex configuration. Within the set T, the set '`convex`' represents all convex configurations. Then, '`most compact`' is the set consisting of every most compact configuration of each chord (in T). Similarly, '`most consonant`' is the set consisting of every most consonant configuration of each chord (in T). The intersection of sets of our interest are given in equation 18.

$$\begin{aligned} a \cup d &= \texttt{convex} \cap \texttt{consonant} \\ b \cup d &= \texttt{compact} \cap \texttt{convex} \\ d \cup c &= \texttt{compact} \cap \texttt{consonant} \\ d &= \texttt{compact} \cap \texttt{convex} \cap \texttt{consonant} \end{aligned} \tag{18}$$

The percentages that we are looking for, are obtained by dividing the number of elements of the sets given in eq. 18, by the number of chords that have a possible convex configuration. Note that the latter value is not equivalent to the number of elements in T, since this set represents the number of configurations instead of the number of chords.

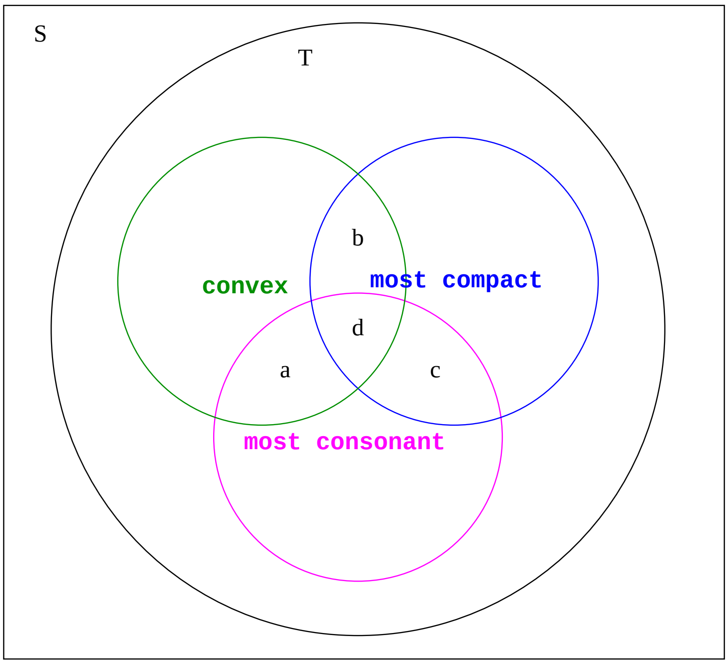


Figure 5: Illustration of overlap of convex, most compact and most consonant configuration when trying to find the preferred intonation of a chord.

There could be more than one convex possibility per set (as is the case with the dominant thirteenth chord in table 3). Also, it is possible that more than one configuration has the same (lowest) value for the compactness or consonance (although this rarely happens). This only means that some solutions are not unique, but since we count the number of chords and not the number of configurations, this does not change the obtained percentages.

We have written a program in Matlab that finds all possible $2, 3$ and 4 note sets in a 9×9 (coordinates run from -4 to 4) 2-dimensional lattice, and calculates for each set 1) whether it has a convex configuration and which configurations are convex, 2) the configuration that is most compact, and 3) the configuration that is most consonant. To distinguish between the configurations, a variable k is used in the same way as introduced in equations 12 and 16 for the expressions for compactness and consonance. The convexity of a set is calculated from the coordinates of the elements in the set; the coordinates change with k as in equation 4. The variable k is varied from -2 to 2. If a wider range was chosen, the obtained points (4) would lie outside the lattice. From short test-runs it was concluded that it is sufficient to work with a 9×9 2-dimensional lattice, a bigger lattice did not significantly change the percentages. This conforms to our intuition, since we found a high correlation between consonance and compactness (table 4), and the more compact a set is, the better it fits into a smaller lattice. One point is chosen in the center $(0, 0)$, so for $n = 2$ only one point is varied, for $n = 3$ two points, and so on. To ensure that some sets are not counted twice, point 3 is varied over the points that point 2 has not been varied over[9] and so on for the points thereafter (point 1 is fixed). The number of possible sets for n points is then calculated as follows: the lattice contains $9 \times 9 = 81$ points. One note is fixed at the origin so there are 80 points left for notes 2 to n to vary over. The number of possibilities[10] to choose $n - 1$ point from 80 points is $\binom{80}{n-1}$. Table 5 shows the number of sets that can be chosen from the lattice for the number of notes varying from 2 to 4. Observing that this number increases very fast as a function of n, one can understand that it is

[9]In pseudo code:

for *i: from* 1 *to total number of points* **do**
    vary point 2
    for *j: from i to total number of points* **do**
        vary point 3
    end
end

[10] $\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$

n	number of possible sets for n points
2	$\binom{80}{1} = 80$
3	$\binom{80}{2} = \frac{80 \cdot 79}{2!} = 3160$
4	$\binom{80}{3} = \frac{80 \cdot 79 \cdot 78}{3!} = 82160$

Table 5: Number of possible sets of n point in a 9×9 lattice with point 1 fixed in the origin.

computationally impossible to go much beyond $n = 4$. These numbers of possible sets would be the numbers that are examined with our algorithm if all these sets have a convex possibility. This turns out not to be the case, so the number of examined sets is reduced. The results of the Matlab program are shown in table 6. In this table it is also indicated how many sets have a convex possibility and are therefore examined. We see that for $n = 2$ all 80 sets have a possible convex configuration. For $n > 2$, this is not the case anymore. For example for $n = 3$, only 1590 sets of the 3160 possible sets have a possible convex configuration.

percentage	n=2	n=3	n=4
compact & consonant	97.5 %	85.4 %	85.6 %
convex & consonant	16.3 %	41.4 %	41.2%
convex & compact	11.3 %	40.8 %	36.0%
compact & convex & consonant	11.3 %	37.3 %	34.1%
number of sets examined	80	1590	14810

Table 6: Results of the percentages as indicated in figure 5.

Observing the results, one can see that the biggest correlation can be found between the most compact and consonant sets, as we expected. The correlation between convexity and the other items is very low for $n = 2$ and gets higher as the number of notes increases. This agrees with our intuition too, since our understanding of the relation between convexity and consonance was through the notion of compactness (see the end of the previous section). When considering only 2 notes, the notion of convexity differs a lot from the notion of compactness, since two notes form a convex set if a line can be drawn between the two notes on which no other notes lie. Therefore it is not easier for two notes to form a convex set if the notes lie close to each other than when the notes lie far from each other, as can be seen from the low correlation between convexity and compactness for $n = 2$. However, for increasing n, the correlation between convexity and compactness increases as well. Note that regions a and b are really small, especially for small n (for $n = 2$, $b = 0$). This means that when the most consonant configurations are also convex, they are most likely to be also the most compact configurations (a); and when the most compact configurations are also convex, they are most likely to be also the most consonant configurations (b). The results from table 4 differ from the results "compact & consonant" in table 6. This difference is due to the difference in sets that is taken into account. In the experiment leading to the results of table 6 only the sets that have a possible convex configuration were taken into account. At the end of section 3.2 we explained why the increase of n causes a decrease of the percentage "compact & consonant". Remarkably, there is no decreasing percentage if n increases from 3 to 4, when sets with a convex possibility are considered (table 6).

5 Musical example

From the previous sections it appears that the notion of compactness may be proposed as a helpful tool when trying to find the right intonation for a chord. We mentioned earlier that the intonation model proposed here only applies to chords in isolation, chords without a musical context. This does not mean however, that this intonation model does not apply to real music, it only means that some additional rules or choices are needed for so-called horizontal intonation (which is referred to as 'melodic intonation' by Fyk [3]) as well. Let us elaborate on this a bit before we give an example of how to tune a chord according to our model.

Consider the note sequence given in figure 6. The (vertical) chords in this musical example can all be tuned according to the compactness intonation model (see table 1 for the preferred intonations). However, the choice

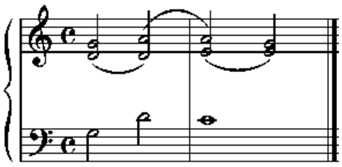


Figure 6: Pitch drift illustrated by $\frac{3}{4} \times \frac{3}{2} \times \frac{3}{5} \times \frac{3}{2} = \frac{81}{80}$. The pitch of the final G will be tuned as 81/80 times the frequency of the first G.

of connecting the intonation from one chord to the other is not obvious, and involves complications. Using the rules for just intonation according to Regener [14] as given in section 1, the horizontal intonation can possibly be worked out as follows. Starting with the first G, in just intonation the D will be tuned a perfect fourth 4/3 below the G. In the adjacent chord, the A will then be tuned as a perfect fifth 3/2 above the D. In the chord thereafter, the C is to be tuned as a major sixth 5/3 below the A. The final G is then tuned as a perfect fifth 3/2 above this C. Comparing the tuning of the first and the last G, we can calculate that the final G is tuned as $\frac{3}{4} \times \frac{3}{2} \times \frac{3}{5} \times \frac{3}{2} = \frac{81}{80}$ of the frequency of the first G. This is called a pitch drift and is a familiar problem related to just intonation. The problem in this example can be solved for example by tuning the subsequent A's in bar 1 and 2 a syntonic comma apart (see Sethares' adaptive tuning example ([20], sec. 8.5.5)). It may be clear from this example that the rules from Regener [14] for just intonation need extension for horizontal intonation as well as for vertical chord intonation. We see however, that the choices of the horizontal intonation do not necessarily change the vertical chord intonation, and therefore the compactness intonation model may appear as a useful intonation model on its own.

To give an example of how to quickly find the preferred intonation of a specific chord, we consider the first chord of bar no. 191 of the Chaconne from the second partita of J.S. Bach (see figure 7). The chord is built from

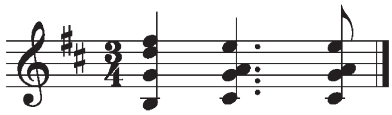
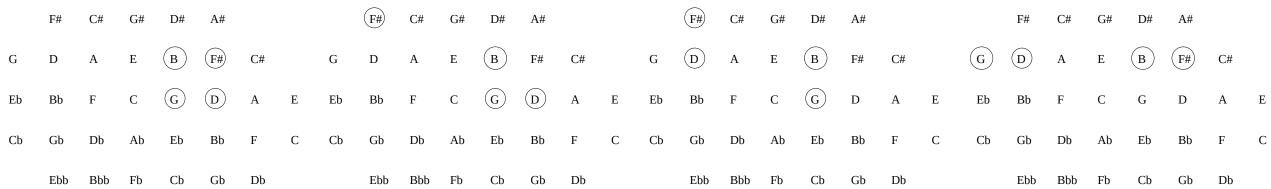


Figure 7: Bar no. 191 of the Chaconne from the second partita of J.S. Bach

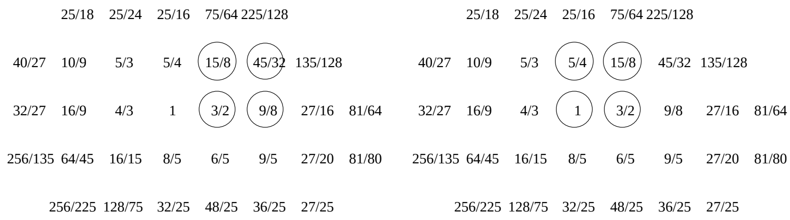
the notes: $B, G, D, F\sharp$. Looking up these notes in the tone space for note names, a lot of different configurations are possible (see figure 8), but one can immediately see which configuration is the most compact. In figure 8 four different configurations are shown, but more configurations are possible. Configuration a is the most compact as can immediately be judged by the eye. This configuration can be projected onto the tone space containing



(a) (b) (c) (d)

Figure 8: Four possible configurations of the chord $B, G, D, F\sharp$.

frequency ratios, to see how to tune the individual notes of the chord, see figure 9a. The ratios indicating the notes in the chord are the frequency ratios in relation to a reference tone C. It might be more useful to know the frequency ratios of the notes in the chord in relation to a reference tone which is one tone of the chord itself. Since the tone-lattice is constructed in such a way that a set of notes can be shifted along the lattice without changing the internal ratios, the chord can just as well be visualized as in figure 9b where the G is chosen to represent the root of the chord. In this way, it is easy to quickly find the preferred intonation of a chord.



(a) (b)

Figure 9: The preferred intonation of the chord $B, G, D, F\sharp$.

6 Concluding remarks

In this paper we have motivated the use of the notions of convexity and compactness as measure of intonation for chords in isolation. We have discussed that most existing consonance measures are difficult to use to decide about different intonations of the same chord. However, the question of 'how to tune chords' is an important one for all musicians that are not limited to a fixed tone instrument. Therefore an easy to use model that selects the best intonation of a chord would be very welcome, and the first step is made in this paper, where we have presented a model for the intonation of chords in isolation. As a measure of consonance, Euler's Gradus function was used for comparison, which, as we motivated, is a representative measure of consonance for this purpose. Although a strong relation was obtained between consonance and both convexity and compactness when tested on some diatonic chords, after investigating all possible chords on a bounded Euler lattice, it turned out that convexity is a poor indication of consonance for chords in isolation. The notion of compactness however, showed a strong relation with consonance for chords with 2, 3 or 4 notes. The high correspondence between Euler's model and the presented compactness model means that the models perform nearly equally well on the task of selecting the most consonant intonation of a chord, with the compactness model having the advantage that it is very simple and intuitive to use, as we have seen in section 5. It is difficult to say something about the cognitive reality of the compactness model, i.e. is every selected intonation really the most consonant version of the chord like humans would judge it? We have explained that Euler's Gradus function does not differ a lot from Helmholtz roughness model in view of the purpose of electing the most consonant tuning of a specific chord. Therefore, in this study, Euler's function represents more than just one consonance model. Assuming that all developed consonance measures try to model cognitive reality as well as possible, we might say that the compactness model corresponds to cognitive reality in a similar way as these other consonance models do, concerning the task of finding the most consonant version of a chord.

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